On Determining the Ranges of the Redundant Quotient Digits of Stefanelli's Division Algorithm

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Abstract— **The upper boundaries of the redundant quotient digits of Stefanelli's division algorithm have been determined. They provide a necessary tool for the cost vs. performance evaluation of the division devices based on Stefanelli's division algorithm.**

The Stefanelli's division algorithm [1] employs the idea of the redundant representation of the quotient. To obtain the quotient $Q \approx C / A$ using this approach, two steps have to be performed. First, the quotient is formulated as a binary number:

$$Q = q_0 2^0 + q_1 2^{-1} + q_2 2^{-2} + \ldots + q_{m-1} 2^{-(m-1)}$$

using the redundant binary digits q_k $(0 \le k \le m-1)$. The redundant binary digits q_k assume either positive or negative integer values to satisfy the following system of algebraic equations:

$$\begin{aligned}
q_0 &= 1 \\
q_1 &= c_2 - a_2 q_0 \\
q_2 &= c_3 - a_2 q_1 - a_3 q_0 \\
q_3 &= c_4 - a_2 q_2 - a_3 q_1 - a_4 q_0 \\
q_4 &= c_5 - a_2 q_3 - a_3 q_2 - a_4 q_1 - a_5 q_0 \\
&\ldots \\
q_{m-1} &= c_m - a_2 q_{m-2} - a_3 q_{m-3} - a_4 q_{m-4} - a_5 q_{m-5} - \ldots - a_m q_0.
\end{aligned} \quad (1)$$

Equations (1) are obtained by assigning the respective sums of partial product array elements with equal weights to the corresponding binary digits of the dividend. The partial product array is obtained by multiplying A by Q. Divisor $A = 0.\ a_1 a_2 \ldots a_n$ and dividend $C = 0.\ c_1 c_2 \ldots c_n$ are assumed to be both positive and normalized binary fractions. In case the quotient has to be formulated with precision equal to the precision of the dividend and the divisor, the number of the redundant quotient digits to be calculated has to be greater than the number of binary bits of the dividend and the divisor. In that case, m-n binary digits of the dividend and the divisor in (1) are set to zero. Once the quotient has been formulated in the redundant binary form using (1), it is converted to the non-redundant binary representation:

$$Q = 2^{0} \cdot q_0^{*} + 2^{-1} \cdot q_1^{*} + 2^{-2} \cdot q_2^{*} + \ldots + 2^{-(n-1)} \cdot q_{n-1}^{*}, \text{ where } q_l^{*} \in \{0,1\} \text{ and } 0 \le l \le n-1.$$

This approach potentially allows substituting the sequential procedure of calculating the quotient digits and the partial reminders [2] with the parallel algebraic summing of some number of operands. This would allow significant reduction of the time latency required to perform the division procedure and to make it comparable with the latency of the multiplication procedure. However, Stefanelli's division algorithm has not been fully researched or evaluated from the prospective of cost vs. performance because the ranges of the redundant quotient digits have not been determined theoretically. That fact makes it difficult to evaluate and to design the division devices based on the discussed approach.

This study provides the method of theoretically determining the ranges of the redundant quotient digits calculated using Stefanelli's division algorithm.

We are going to determine the conditions that allow the redundant quotient digit q_r to reach its maximum for even values of r and its minimum for odd values of r. Because $a_1 = c_1 = 1$, the redundant quotient digit q_0 is always equal to one. Therefore, the theoretical analysis of (1) starts with the equation for q_1. It is obvious that q_1 assumes the minimum value only if $c_2 = 0$ and $a_2 = 1$. The third equation of (1) for determining the value of q_2 demonstrates that the necessary condition for the maximum of q_2 is the minimum of q_1 while $a_2 = 1$. The sufficient condition for the maximum of q_2 requires satisfying the following relationships: $q_1 = min\ q_1$ while $a_2 = 1$, $c_3 = 1$, and $a_3 = 0$. The fourth equation of (1) allows us to determine the condition for $q_3 = min\ q_3$. It is obvious that in order to achieve this condition, c_4 has to be equal to 0 and a_4 has to be equal to 1. Because the quotient digits q_2 and q_1 are included in the fourth equation of (1) with the negative coefficients equal to $-a_2$ and $-a_3$ respectively, it is possible to assume that the minimum of q_3 is achieved in the case where both q_2 and q_1 assume their maximum (positive) values. However, as it was obvious from the previous discussion, q_2 and q_1 cannot possibly assume their respective maximum values at the same time because the maximum value of q_2 is achieved while q_1 assumes the minimum value. Taking into account the fact that according to [1] the allowed range of q_r increases with the increase of r, it is conjectured that the value of q_{r-1} influences the value of q_r to a higher degree than the value of q_{r-2}. Therefore, it is conjectured that the quotient digit q_3 assumes its minimum value under the following conditions: $q_2 = max\ q_2$ while $a_2 = 1$ and $q_1 = min\ q_1$ while $a_3 = 0$, and $c_4 = 0$ and $a_4 = 1$. As above, it is possible to establish that q_4 assumes its maximum value under the following conditions: $q_3 = min\ q_3$ while $a_2 = 1$, and $q_2 = max\ q_2$ while $a_3 = 0$, and $q_1 = min\ q_1$ while $a_4 = 1$ and $c_4 = 0$, $c_5 = 1$, and $a_5 = 0$.

Based upon these considerations, it is possible to determine the respective conditions for the maximum or the minimum of other redundant quotient digits q_r. For even values of k, the maximum value of q_k is achieved in the case $q_{k-1} = min\ q_{k-1}$ while $a_2 = 1$, $q_{k-2} = max\ q_{k-2}$ while $a_3 = 0$, $q_{k-3} = min\ q_{k-3}$ while $a_4 = 1, \ldots, q_2 = max\ q_2$ while $a_{k-1} = 0$, and $q_1 = min\ q_1$ while $a_k = 1$, and $c_{k+1} = 1$, $a_{k+1} = 0$, i.e. when the dividend $C =$

$\underbrace{0.101010...101}_{k+1}xxx...x$ and the divisor $A = \underbrace{0.110101...010}_{k+1}xxx...x$, where $x \in \{0,1\}$. For the odd values of k, the minimum value of q_k is achieved in the case $q_{k-1} = max\ q_{k-1}$ while $a_2 = 1$, $q_{k-2} = min\ q_{k-2}$ while $a_3 = 0$, $q_{k-3} = max\ q_{k-3}$ while $a_4 = 1$, ... , $q_2 = max\ q_2$ while $a_{k-1} = 1$, and $q_1 = min\ q_1$ while $a_k = 0$, and $c_{k+1} = 0$, and $a_{k+1} = 1$, i.e. when the dividend $C = \underbrace{0.101010...010}_{k+1}xxx...x$ and the divisor $A = \underbrace{0.110101...101}_{k+1}xxx...x$. The values of the dividend C and the divisor A that allow the redundant quotient digits to achieve the respective maximum values for even r and the minimum values for odd r shall be called *critical codes*. If we substitute the above critical code for the dividend C and the critical code for the divider A to (1) and solve it for the values of $q_r^{'}$, then it is easy to see that the corresponding values of $q_r^{'}$ assume the values of the sign-alternating Fibonacci sequence [3]:

$$q_r^{'} = (-1)^r\ U_{r+1}, \qquad (2)$$

where U_{r+1} is $(r+1)th$ number of the Fibonacci sequence.

Taking into account Binet's formula [3], equation (2) can be represented as follows:

$$q_r^{'} = (-1)^r\ [((1+\sqrt{5})/2)^{r+1} - ((1-\sqrt{5})/2)^{r+1}\ ] / \sqrt{5} \qquad (3)$$

The above expression allows us to determine the maximum values of the redundant quotient digits q_r for even values of r and the minimum values of q_r for odd values of r.

To determine the conditions that allow the redundant quotient digits q_r to reach the maximum values for odd values of r and the minimum values for even values of r we will use the following considerations. Assume that $q_1 = max\ q_1$. In this case, according to the third equation of (1) for determining the value of q_2, the necessary condition for the minimum of q_2 is the following: $q_1 = max\ q_1$ while $a_2 = 1$. However, as the second equation of (1) indicates, q_1 cannot possibly assume its maximum value equal to 1 while $a_2 = 1$. Therefore, to make sure that the condition of $q_2 = min\ q_2$ does not contradict the condition of $q_1 = max\ q_1$, we will assume that the quotient digit q_0 can assume the values of both 1 and 0. In this case, by selecting $q_0 = 0$, we ensure that $q_1 = max\ q_1 = 1$ while $c_2 = 1$ and $a_2 = 1$. In this case, the necessary condition of $q_2 = min\ q_2$ described above does not contradict the condition of $q_1 = max\ q_1$. The sufficient condition for the minimum of q_2 requires satisfying the following relationships: $q_1 = max\ q_1$ while $a_2 = 1$ and $c_3 = 0$. The value of a_3 is not included in the condition of $q_2 = min\ q_2$ because in this case the value of a_3 has a multiplier q_0 that is equal to 0.

Based on the fourth equation of (1), the maximum of the redundant quotient digit q_3 can be achieved if the redundant quotient digits q_2 and q_1 assume their respective minimum values. However, based on the previous considerations, q_2 and q_1 cannot possibly assume the respective minimum values at the same time. Therefore, assuming that the value of q_2 influences the value of q_3 to a higher degree than the value of q_1, it is conjectured that q_3 assumes its maximum value under the following conditions: $q_2 = min\ q_2$ while $a_2 = 1$.

Obviously, in that case, a_3 has to be set to 0, because $q_1 = max\ q_1$. The sufficient condition for the maximum of q_3 requires satisfying the following relationships: $q_2 = min\ q_2$ while $a_2 = 1$, $q_1 = max\ q_1$ while $a_3 = 0$, and $c_4 = 1$. The value of a_4 is not included in the condition of $q_3 = max\ q_3$ because in this case the value of a_4 has a multiplier q_0 which is equal to 0.

Similar to the above, it is possible to establish that q_4 assumes its minimum value under the following conditions: $q_3 = max\ q_3$ while $a_2 = 1$, and $q_2 = min\ q_2$ while $a_3 = 0$, and $q_1 = max\ q_1$ while $a_4 = 1$, and $c_5 = 0$.

Using this approach, it is possible to determine that, for odd values of k, the maximum value of q_k is achieved in the case $q_{k-1} = min\ q_{k-1}$ while $a_2 = 1$, $q_{k-2} = max\ q_{k-2}$ while $a_3 = 0$, $q_{k-3} = min\ q_{k-3}$ while $a_4 = 1, \ldots,$ $q_2 = min\ q_2$ while $a_{k-1} = 1$, and $q_1 = max\ q_1$ while $a_k = 0$, and $c_{k+1} = 1$, i.e. when the dividend $C = \underbrace{0.101010...101}_{k+1}xxx...x$ and the divisor $A = \underbrace{0.110101...010}_{k}xxx...x$, and $q_0 = 0$. For even values of k, the minimum value of q_k is achieved when $q_{k-1} = max\ q_{k-1}$ while $a_2 = 1$, $q_{k-2} = min\ q_{k-2}$ while $a_3 = 0$, $q_{k-3} = max\ q_{k-3}$ while $a_4 = 1, \ldots,$ $q_2 = min\ q_2$ while $a_{k-1} = 0$, and $q_1 = max\ q_1$ while $a_k = 1$, and $c_{k+1} = 0$, i.e. when the dividend $C = \underbrace{0.101010...010}_{k+1}xxx...x$ and divisor $A = \underbrace{0.110101...101}_{k}xxx...x$ and $q_0 = 0$. If we substitute the respective values for the dividend C and the divisor A to (1) and solve it for the values of q_r'', then it is easy to see that the corresponding values of q_r'' assume the values of the sign-alternating Fibonacci sequence [3]:

$$q_r'' = (-1)^{r+1}\ U_r = (-1)^{r+1}\ [((1+\sqrt{5})/2)^r - ((1-\sqrt{5})/2)^r] / \sqrt{5}, \qquad (4)$$

where U_r is rth number of the Fibonacci sequence.

Note that the maximum and the minimum values of the redundant quotient digits q_r for the respective odd and even values of r have been determined by (4) while assuming that q_0 can assume both the values of 0 and 1. This means that the absolute values of these maximum and minimum values according to (4) will be greater than the actual maximum and minimum values of the respective redundant quotient digits q_r and we can assume that we can satisfy (1) based on $q_0 = 1$. Based on that consideration and also based on the fact that $U_r < U_{r+1}$ for $r > 1$, it is easy to see that

$$max\ |q_r| = U_{r+1} = [((1+\sqrt{5})/2)^{r+1} - ((1-\sqrt{5})/2)^{r+1}] / \sqrt{5} \qquad (5)$$

Thus, the equation (5) allows us to determine the upper boundaries of the absolute values of the redundant quotient digits q_r. In particular, for n most significant redundant quotient digits, the upper boundaries are equal to the maximum absolute values of the respective redundant quotient digits. For the case of m-n least significant quotient digits, the actual maximum absolute values of the redundant quotient digits are smaller than the upper boundaries determined according to (5). The reason for this is that the equation (5) determines the boundaries of the respective maximum absolute values in assumption that all m bits of dividend and divisor are set to the values of the respective “critical code” as they were defined earlier whereas in reality m-n least significant bits of the dividend and the divisor are set to *0.*

The computer modeling of Stefanelli’s division algorithm confirmed the correctness of all the assumptions of this study. It also confirmed the fact that the maximum absolute values of the redundant quotient digits are achieved if dividend C and divisor A are set to the values of the respective “critical codes.”

The table below presents some of the results of computer simulation which confirm the fact that the maximum absolute values of the redundant quotient digits q_r are equal to the corresponding Fibonacci numbers [3].

Redundant quotient digits	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
$max\ q_r$	1	1	2	2	5	4	13	8	34	16
$min\ q_r$	1	-1	-1	-3	-3	-8	-5	-21	-12	-55
$max\ \lvert q_r \rvert$	1	1	2	3	5	8	13	21	34	55

The results of the theoretical analysis performed in this study provide the necessary tool for a cost vs. performance evaluation of the division devices based on Stefanelli’s division algorithm and for comparing it with other known division procedures.

REFERENCES

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